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Some First Passage Time Problems for Shot Noise Processes

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It has recently been shown that the first passage time problem for a certain class of one-dimensional processes that includes shot noise can be formulated in terms of a set of integral equations. These are found by exact enumeration of all possible trajectories. We show that the equations can be found by more direct means for processes described by the evolution equation $\dot{x}(t) = f(x) + n(t)$, where $n(t)$ is time-localized shot noise.

KEY WORDS: First passage times; shot noise; regenerative processes; renewal theory.

1. INTRODUCTION

The theory of first passage times for time-homogeneous Markov processes in the presence of absorbing states is well understood, although technical problems may remain in particular applications.[1,2] The corresponding theory for non-Markovian processes is in a much more fragmented state because of the large number of possible models. One such class of non-Markovian models for which some progress has been possible is that in which the noise is dichotomous. This has been studied by a number of authors, the early analyses being valid for telegraph signal noise,[3-5] i.e., the time points at which the noise changes sign are described by a Markov process. Recently Masoliver *et al.*[6] developed a formalism for calculating statistical properties of the first passage time (FPT) for a particular class of

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one-dimensional processes driven by non-Markovian dichotomous noise. The methodology of that reference requires an exact enumeration of all possible events and is therefore somewhat complicated. More recently Weiss *et al.*[7] have shown that a complete enumeration of events is unnecessary and the integral equation derived in Ref. 6 can be found directly. This more direct approach allows one to extend the analysis to a larger class of multivalued noise models.

A second class of non-Markovian processes for which some results on statistical properties of FPTs can be found is that of shot noise.[8] Masoliver has shown that one can use the technique of event enumeration to derive an integral equation for the mean first passage time out of the interval, $I = (0, L)$. In his analysis it is assumed that $x(t)$ evolves deterministically in the absence of shot noise impulses.

In the present paper we assume that $x(t)$ is the solution to the one-dimensional dynamical equation

$$\dot{x}(t) = f(x) + n(t) \tag{1}$$

where $n(t)$ is assumed to be the shot noise process

$$n(t) = \sum_{i=1}^{\infty} \gamma_i \, \delta(t - t_i) \tag{2}$$

where the γ_i and $\Delta_i = t_i - t_{i-1}$, $i = 1, 2,...$, are assumed to be independent, identically distributed random variables and $\delta(t)$ is a delta function. Equation (1) is to be solved subject to the initial condition $x(0) = x_0$, our object being to calculate statistical properties of the FPT to one of the boundaries of I. We will show that the enumeration method used by Masoliver[8] can be replaced by a more direct method of solution. It will be shown that the expression for the survival probability can be written in terms of a function $v(x, t)$, to be defined. This function, together with a second function that enters the problem in a natural way, is shown to be found as the solution to a coupled set of integral equations. The Laplace transforms of these functions satisfy a somewhat simpler set of equations, which is convenient for the calculation of moments of the FPT. In special cases these equations can be transformed into a single second-order differential equation together with appropriate boundary conditions, but in general one is left with the coupled integral equations, which are then to be analyzed numerically.

2. ANALYSIS

The reason that one can give a relatively simple analysis of the FPT problem for dynamical systems with shot noise of the form given in Eq. (2)

is that the impulses composing the shot noise are assumed to occur instantaneously. Thus, it is possible to decompose the resulting dynamical evolution into intervals of impulse-free motion and instantaneous jumps due to the impulses. It is considerably more complicated to deal with the effects of impulses that are nonlocal in time. This more general case appears to pose a mathematical problem of great difficulty that we have been unable to solve.. In our analysis of the simpler case the instants at which impulses occur, the $\{t_i\}$, $i = 1, 2,...$, will be regarded as constituting a regenerative process in the sense of Smith,[9] which suggests the form of the analysis that follows.

Let us first define the functions needed to describe our problem as well as the mathematical assumptions used in the solution.

1. The probability density of the impulse amplitudes γ_i will be denoted by $h(\gamma)$. In the most general analysis no restrictions need be placed on the sign or magnitude of the γ_i for the following analysis.

2. The common probability density for the inter-impulse times, the Δ's, will be denoted by $\psi(\Delta)$, so that the probability that $t_i - t_{i-1} > \Delta$ is

$$\Psi(\Delta) = \int_{\Delta}^{\infty} \psi(u)\, du \tag{3}$$

3. The impulse-free evolution will be described by a known function $x(t) = X(t \mid x_0)$, where $x_0 = x(0)$. This function is the solution to $\dot{x} = f(x)$ and will be assumed to be a monotonic in t, that is, $f(x)$ will be assumed to have the same sign for all x in I. For the sake of definiteness $f(x)$ will be chosen to be negative, so that $\dot{x} < 0$ throughout the interval. This implies that any exit from I that occurs between two successive impulses will always occur at the lower boundary, $x = 0$.

4. It will be convenient to define a travel time between two points a and b, both in I. Since the dynamical system is autonomous, i.e., the function $f(x)$ does not contain the time except implicitly through x, the travel time between points a and b can be calculated as

$$\begin{aligned} t(a, b) &= \int_b^a du/|f(u)| \qquad &\text{for} \quad a > b \\ t(a, b) &= 0 &\text{for} \quad a < b \end{aligned} \tag{4}$$

Since, as we have noted, the times at which impulses occur constitute a regenerative process, a calculation of properties of the FPT requires only that we know the state of the system at the impulse times t_i. Two functions are required to specify these properties. These are defined at the jump points only and will be denoted by $u(x, t \mid x_0)$ and $v(x, t \mid x_0)$. The two functions are defined by

$$\begin{aligned} u(x, t \mid x_0)\, dx &= \Pr\{x < \lim_{\Delta t \to 0} x(t-\Delta t) < x + dx \mid x(0) = x_0\} \\ v(x, t \mid x_0)\, dx &= \Pr\{x < \lim_{\Delta t \to 0} x(t+\Delta t) < x + dx \mid x(0) = x_0\} \end{aligned} \tag{5}$$

The two functions differ because of the impulse at time t, but they are not independent, since $v(x, t \mid x_0)$ can be expressed in terms of $u(x, t \mid x_0)$ and $h(y)$, as will be seen shortly. In addition to the functions u and v, it is necessary for us to define an initial function $u_1(x, t_1 \mid x_0)$ analogous to but generally different from $u(x, t \mid x_0)$, since all of the t_i have a preceding impulse except t_1.

Let us first express the probability $S(t \mid x_0)$ that $x(t)$ is still in I at time t, given the initial position x_0, in terms of the functions $u(x, t \mid x_0)$, $u_1(x, t \mid x_0)$, and $v(x, t \mid x_0)$ defined earlier. In our final step we derive the equations satisfied by these functions. The expression for $S(t \mid x_0)$ is

$$\begin{aligned} S(t \mid x_0) &= \Psi(t) \int_0^L dx\, \delta(x - X(t \mid x_0)) \\ &\quad + \int_0^L dx \int_0^L dy \int_0^t v(y, \tau \mid x_0)\, \delta[x - X(t-\tau \mid y)]\, \Psi(t-\tau)\, d\tau \end{aligned} \tag{6}$$

The first term is the contribution from the time before the first impulse, and the second the contribution from times after the occurrence of at least one impulse. As a final step we need to specify the set of integral equations satisfied by the functions u, x, and v. These are found as the solution to

$$\begin{aligned} u(x, t) &= u_1(x, t) + \int_0^t d\tau \int_0^L dy\, v(y, \tau)\, \delta[x - X(t-\tau \mid y)]\, \psi(t-\tau) \\ v(x, t) &= \int_0^L dy\, u(y, \tau)\, h(x-y), \qquad 0 < x < L \end{aligned} \tag{7}$$

in which we have suppressed the argument x_0 which should appear in all of the u's and v's. One can also combine the pair of integral equations into a single one involving a double integral. The function $u_1(x, t)$ can be expressed in terms of functions defined earlier as

$$u_1(x, t \mid x_0) = \delta[x - X(t \mid x_0)]\, \psi(t) \tag{8}$$

Equations (6)–(8) give a complete formulation of the FPT problem, since the FPT moments can be expressed as

$$\langle t_n(x_0) \rangle = n \int_0^\infty \tau^{n-1} S(\tau \mid x_0)\, d\tau \tag{9}$$

In the most general cases Eqs. (6)–(8) can be used as the basis of a numerical solution of the FPT problem. It will prove convenient in what follows to use the Laplace-transformed version of this set of equations, since the FPT moments are simply expressed in terms of these. If we denote the Laplace transform of a function by the same function with a caret, e.g., $\{S(t \mid x_0)\} = \hat{S}(s \mid x_0)$, then Eq. (9) is seen to be equivalent to

$$\langle t_n(x_0)\rangle = (-1)^n n\, d^{n-1}\hat{S}/ds^{n-1}\big|_{s=0} \tag{10}$$

The Laplace transforms of the components of Eq. (7) are

$$\begin{aligned} \hat{u}(x, s) &= \hat{u}_1(x, s) - \frac{1}{f(x)} \int_x^L dy\, \hat{v}(y, s)\, \hat{\psi}(t(x, y))\, e^{-st(x,y)} \\ \hat{v}(x, s) &= \int_0^L dy\, \hat{u}(y, s)\, h(x-y) \end{aligned} \tag{11}$$

One can find an integral equation for the mean FPT by taking the Laplace transform of Eq. (6) and making use of Eq. (10). Rather than doing this in complete generality, let us restrict ourselves to the case of an exponential waiting time density

$$\psi(t) = \lambda e^{-\lambda t} \tag{12}$$

and define $v(y) \equiv \hat{v}(y, 0)$. We then find that $\langle t_1(x_0)\rangle$ is the solution to

$$\langle t_1(x_0)\rangle = -\int_0^{x_0} dx \frac{e^{-\lambda t(x,x_0)}}{f(x)} - \int_0^L dx \int_x^L dy\, e^{-\lambda t(x,y)} v(y) \tag{13}$$

where $t(x, y)$ is the travel time defined in Eq. (4). Let us specialize the problem even more by assuming that the impulses can only be positive and are described by a negative exponential density,

$$\begin{aligned} h(x) &= \gamma e^{-\gamma x}, && x > 0 \\ &= 0, && x < 0 \end{aligned} \tag{14}$$

Then the system of coupled integral equations in Eq. (11) can be reduced to a second-order inhomogeneous differential equation for the function $v(x)$ needed to evaluate the expression for $\langle t_1(x_0)\rangle$ in Eq. (13). The result is

$$\frac{d^2 v(x)}{dx^2} + \left[\gamma + \frac{f'(x) + \lambda}{f(x)}\right]\frac{dv}{dx} + \frac{\gamma f'(x)\, v(x)}{f(x)} = \lambda\gamma\, \delta(x - x_0) \tag{15}$$

which is to be solved subject to the boundary conditions

$$v(0)=0, \qquad dv/dx\,|_{x=L}=-\gamma v(L) \tag{16}$$

There are only a few cases in which these equations can be solved in closed form. One of these is for the case of uniform motion $x(t)=-t$, in the absence of impulses. If we set $\lambda=\gamma=1$ for simplicity, then we find that the function $v(x)$ satisfying Eqs. (15) and (16) is where $\theta(x)$ is the Heaviside step function. In this case one finds $\langle t_1(x_0)\rangle$ to be

$$\langle t_1(x_0)\rangle = -\frac{x_0^2}{2}+\left(\frac{1+L+L^2/2}{1+L}\right)x_0 \tag{18}$$

It is also possible, using present methods, to analyze the case in which $f(x)=0$, so that in the absence of impulses the system remains stationary. However, generalization of the theory to analyze FPT properties of the nonautonomous dynamical system whose evolution is described by

$$\dot{x}=f(x,t)+n(t) \tag{19}$$

seems quite difficult.

Finally, we comment that an extension of our formalism allows one to treat the case in which the impulse-free motion is itself a random process. This generalization is under investigation.

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